16 The Side-Angle-Side Axiom

1. (a) Let $\overline{MN}$ and $\overline{PQ}$ denote two given line segments. Explain what it mean that $\overline{MN} \cong \overline{PQ}$.

(b) Let $\measuredangle ABC$ and $\measuredangle DEF$ denote given angles. Explain, what it mean that $\measuredangle ABC \cong \measuredangle DEF$.

Intuitively, two figures are congruent if one can be "picked up and laid down exactly on the other" so that the two coincide.

Convention. In $\triangle ABC$, if there is no confusion, we will denote $\measuredangle CAB$ by $\measuredangle C$, $\measuredangle ABC$ by $\measuredangle B$ and $\measuredangle BCA$ by $\measuredangle C$.

Definition (congruence, congruent triangles)
Let $\triangle ABC$ and $\triangle DEF$ be two triangles in a protractor geometry and let $f : \{A,B,C\} \to \{D,E,F\}$ be a bijection between the vertices of the triangles. f is a congruence iff

$$\overline{AB} = \overline{f(A)f(B)}, \qquad \overline{BC} = \overline{f(B)f(C)}, \qquad \overline{CA} = \overline{f(C)f(A)},$$

$$\measuredangle A \cong \measuredangle f(A), \qquad \measuredangle B \cong \measuredangle f(B) \qquad \text{and} \qquad \measuredangle C \cong \measuredangle f(C).$$

Two triangles, $\triangle ABC$ and $\triangle DEF$, are congruent if there is a congruence $f : \{A,B,C\} \to \{D,E,F\}$. If the congruence is given by $f(A) = D$, $f(B) = E$, and $f(C) = F$, then we write $\triangle ABC \cong \triangle DEF$.

2. Prove that congruence is an equivalence relation on the set of all triangles in a protractor geometry.

The fundamental question of this section is: How much do we need to know about a triangle so that it is determined up to congruence?

Suppose that we are given $\triangle ABC$ and a ray $\overrightarrow{EX}$ which lies on the edge of a half plane H_1. Then we can construct the following by the Segment Construction Theorem and the Angle Construction Theorem
(a) A unique point $D \in \overrightarrow{EX}$ with $\overline{BA} \cong \overline{ED}$;
(b) A unique ray $\overrightarrow{EY}$ with $Y \in H_1$ and $\measuredangle ABC \cong \measuredangle XEY$;
(c) A unique point $F \in \overrightarrow{EY}$ with $\overline{BC} \cong \overline{EF}$.

Is $\triangle ABC \cong \triangle DEF$? Intuitively it should be (and it will be if SAS is satisfied). However, since we know nothing about the rulers for $\overleftrightarrow{DF}$ and $\overleftrightarrow{AC}$, we have no way of showing that $\overline{AC} \cong \overline{DF}$. In fact next example will show that $\overline{AC}$ need not be congruent to $\overline{DF}$.

3. In the Taxicab Plane let $A(1,1)$, $B(0,0)$, $C(-1,1)$, $E(0,0)$, $X(3,0)$, and let H_1 be the half plane above the x-axis. Carry out the construction outlined above and check to see whether or not $\triangle ABC$ is congruent to $\triangle DEF$.

[Example 6.1.1, page 126]

Definition (Side-Angle-Side Axiom (SAS))
A protractor geometry satisfies the Side-Angle-Side Axiom (SAS) if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $\overline{AB} \cong \overline{DE}$, $\measuredangle B \cong \measuredangle E$ and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

Definition (neutral or absolute geometry)
A neutral geometry (or absolute geometry) is a protractor geometry which satisfies SAS.

Proposition (Euclidean Law of Cosines). Let $c(\theta)$ be the cosine function as developed in Section 15. Then for any $\triangle PQR$ in the Euclidean Plane $d_E(P,R)^2 = d_E(P,Q)^2 + d_E(Q,R)^2 - 2d_E(P,Q)d_E(Q,R)\,c(m_E(\measuredangle PQR))$.

Proposition. The Euclidean Plane $\mathcal{E}$ satisfies SAS.

4. Prove the above Proposition.

[Proposition 6.1.3, page 128]

Proposition. The Poincaré Plane $\mathbb{H}$ is a neutral geometry.

Definition (isosceles triangle, scalene triangle, equilateral triangle, base angles) A triangle in a protractor geometry is isosceles if (at least) two sides are congruent. Otherwise, the triangle is scalene. The triangle is equilateral if all three sides are congruent. If $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{BC}$, then the base angles of $\triangle ABC$ are $\measuredangle A$ and $\measuredangle C$.

Our first application of SAS is the following theorem on isosceles triangles. The Latin name (literally "the bridge of asses") refers to the complicated figure Euclid used in his proof, which looked like a bridge, and to the fact that only someone as dull as an ass would fail to understand it.

Theorem. **(Pons Asinorum)**. In a neutral geometry, the base angles of an isosceles triangle are congruent.

5. Prove the above Theorem.

[Theorem 6.1.5, page 129]

6. Let $\triangle ABC$ be an isosceles triangle in a neutral geometry with $\overline{AB} \cong \overline{CA}$. Let M be the midpoint of $\overline{BC}$. Prove that $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$.

7. Prove that in a neutral geometry every equilateral triangle is **equiangular**; that is, all its angles are congruent.

8. Show that if $\triangle ABC$ is a triangle in the Euclidean Plane which has a right angle at C then $(AB)^2 = (AC)^2 + (BC)^2$.

9. Let $\triangle ABC$ be a triangle in the Euclidean Plane with $\measuredangle C$ a right angle. If $m_E(\measuredangle B) = \theta$ prove that $c(\theta) = BC/AB$ and $s(\theta) = AC/AB$.

10. Let $\square ABCD$ be a quadrilateral in a neutral geometry with $\overline{CD} \cong \overline{CB}$. If $\overrightarrow{CA}$ is the bisector of $\measuredangle DCB$ prove that $\overline{AB} \cong \overline{AD}$.

11. Let $\square ABCD$ be a quadrilateral in a neutral geometry and assume that there is a point $M \in \overline{BD} \cap \overline{AC}$. If M is the midpoint of both $\overline{BD}$ and $\overline{AC}$ prove that $\overline{AB} \cong \overline{CD}$.

12. Suppose there are points A, B, C, D, E in a neutral geometry with $A-D-B$ and $A-E-C$ and A, B, C not collinear. If $\overline{AD} \cong \overline{AE}$ and $\overline{DB} \cong \overline{EC}$ prove that $\measuredangle EBC \cong \measuredangle DCB$.

17 Basic Triangle Congruence Theorems

Definition. **(Angle-Side-Angle Axiom (ASA))** A protractor geometry satisfies the Angle-Side-Angle Axiom (ASA) if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $\measuredangle A \cong \measuredangle D$, $\overline{AB} \cong \overline{DE}$, and $\measuredangle B \cong \measuredangle E$, then $\triangle ABC \cong \triangle DEF$.

Theorem. A neutral geometry satisfies ASA.

1. Prove the above Theorem.

[Theorem 6.2.1, page 131]

Theorem. **(Converse of Pons Asinorum)**. In a neutral geometry, given $\triangle ABC$ with $\measuredangle A \cong \measuredangle C$, then $\overline{AB} \cong \overline{CB}$ and the triangle is isosceles.

2. Prove the above Theorem.

3. Prove that in a neutral geometry every equiangular triangle is also equilateral.

Definition. **(Side-Side-Side Axiom (SSS))** A protractor geometry satisfies the Side-Side-Side Axiom (SSS) if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.

Theorem. A neutral geometry satisfies SSS.

4. Prove the above Theorem.

[Theorem 6.2.3, page 132]

In one of earlier sections we showed that PSA and PP are equivalent axioms: if a metric geometry satisfies one of them then it also satisfies the other. A similar situation is true for SAS and ASA. We already know that SAS implies ASA. The next theorem gives the converse.

Theorem. If a protractor geometry satisfies ASA then it also satisfies SAS and is thus a neutral geometry.

5. Prove the above Theorem.

Theorem. In a neutral geometry, given a line ℓ and a point $B \notin \ell$, then there exists at least one line through B perpendicular to ℓ.

6. Prove the above Theorem.

[Theorem 6.2.5, page 133]

7. In a neutral geometry, given $\triangle ABC$ with $\overline{AB} \cong \overline{BC}$, $A-D-E-C$, and $\measuredangle ABD \cong \measuredangle CBE$, prove that $\overline{DB} \cong \overline{EB}$.

8. In a neutral geometry, given $\triangle ABC$ with $A-D-E-C$, $\overline{AD} \cong \overline{EC}$, and $\measuredangle CAB \cong \measuredangle ACB$, prove that $\measuredangle ABE \cong \measuredangle CBD$.

9. In a neutral geometry, given $\square ABCD$ with $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, prove that $\measuredangle A \cong \measuredangle C$ and $\measuredangle B \cong \measuredangle D$.

10. In a neutral geometry, given $\triangle ABC$ with $A-D-B$, $A-E-C$, $\measuredangle ABE \cong \measuredangle ACD$, $\measuredangle BDC \cong \measuredangle BEC$, and $\overline{BE} \cong \overline{CD}$, prove that $\triangle ABC$ is isosceles.